Symmetric Unions and the Jones Polynomial

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Background

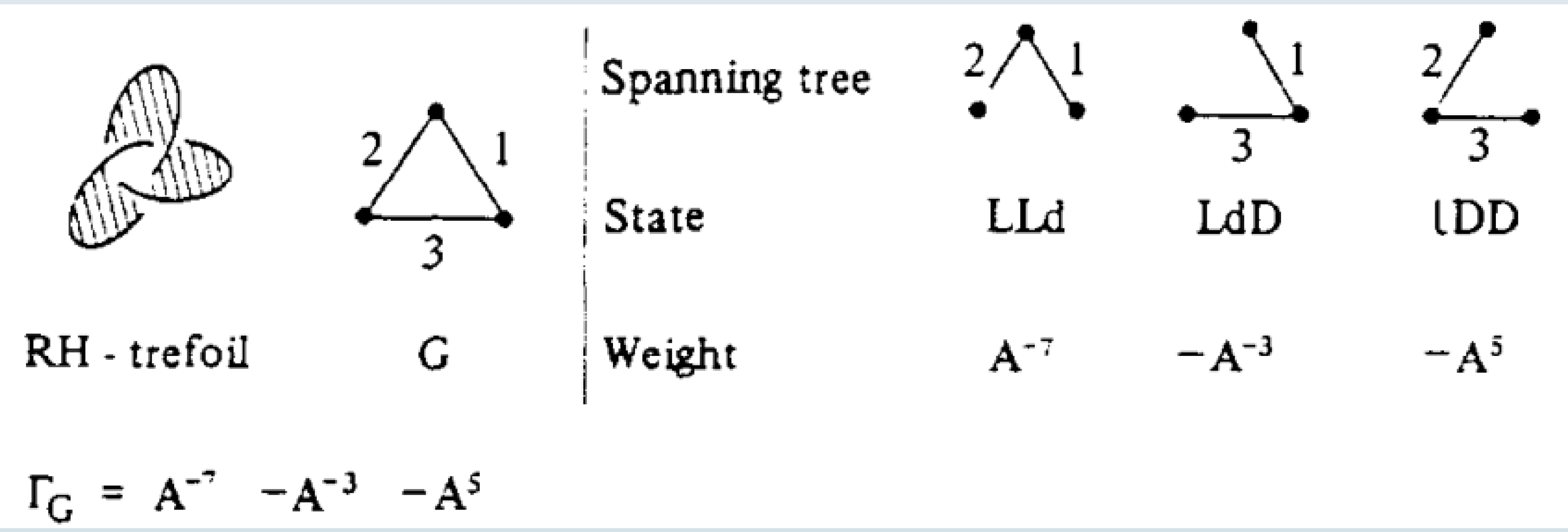
Symmetric unions are formed by taking the union of a knot diagram and its mirror image, and then placing crossings on the axis of symmetry. A knot invariant is a quantity placed on a knot that is unchanged when Reidemeister moves are applied to it. The invariant aimed for here was a polynomial, but invariants may be even simpler than this. Famous invariants include the Jones polynomial and the Alexander polynomial. In a group action, a group permutes the elements of a set X. The essence of a quotient is that it glues a topological space onto itself by a group action.

First Line of Inquiry

We began by looking into the Tutte polynomial $\chi_G(x, y)$ of the graph G and the concepts of internal and external activities of a graph as outlined by Morwen Thistlethwaite. The Tutte polynomial $\chi_G(x, y)$ is the polynomial $\sum_{T \epsilon G} x^r y^s$, where the sum is taken over all the spanning trees T of G, and r, s are respectively the internal and external activities of T. Following this, we looked at the Gamma polynomial Γ_G associated to a spanning tree based on its internal and external activities. Γ_G is formed by categorising activity into one of eight possible states, given in the table below.

For our main point of inquiry, we asked the following question:

Q: For each $e \epsilon T$ in a spanning tree T for a graph G, is $cut(T, e) = \rho(cut(\rho(T), \rho(e)))$ where ρ is the reflection in the axis of symmetry?



(Thistlethwaite, 1987, fig.9)

Table 2

State of e_j	L	D	ℓ	d	$\bar{L}$	$\bar{D}$	$\bar{\ell}$	$\bar{d}$
μ_{ij}	$-A^{-3}$	A	$-A^3$	A^{-1}	$-A^3$	A^{-1}	$-A^{-3}$	A

(Thistlethwaite, 1987, p.305)

Aims of the Investigation

This project focused on the 'symmetric' Reidemeister moves coined by Eisermann and Lamm, and their on and off-axis variations, as well as off-axis variations of the classical Reidemeister moves, as shown below. Drawing on previous research by Eisermann and Lamm, Louis Kauffman, and Morwen Thistlethwaite, this project aimed to examine possibilities of invariants on the symmetric union.

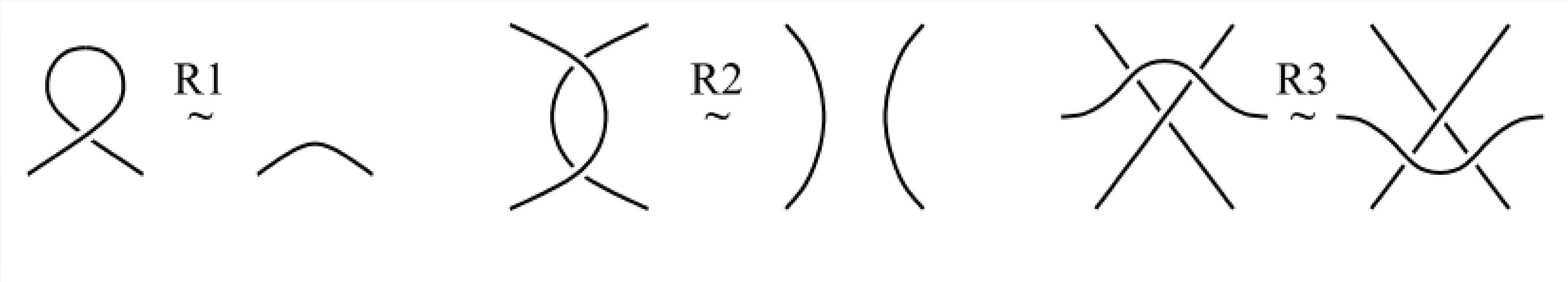


Fig. 4. The classical Reidemeister moves (off the axis).

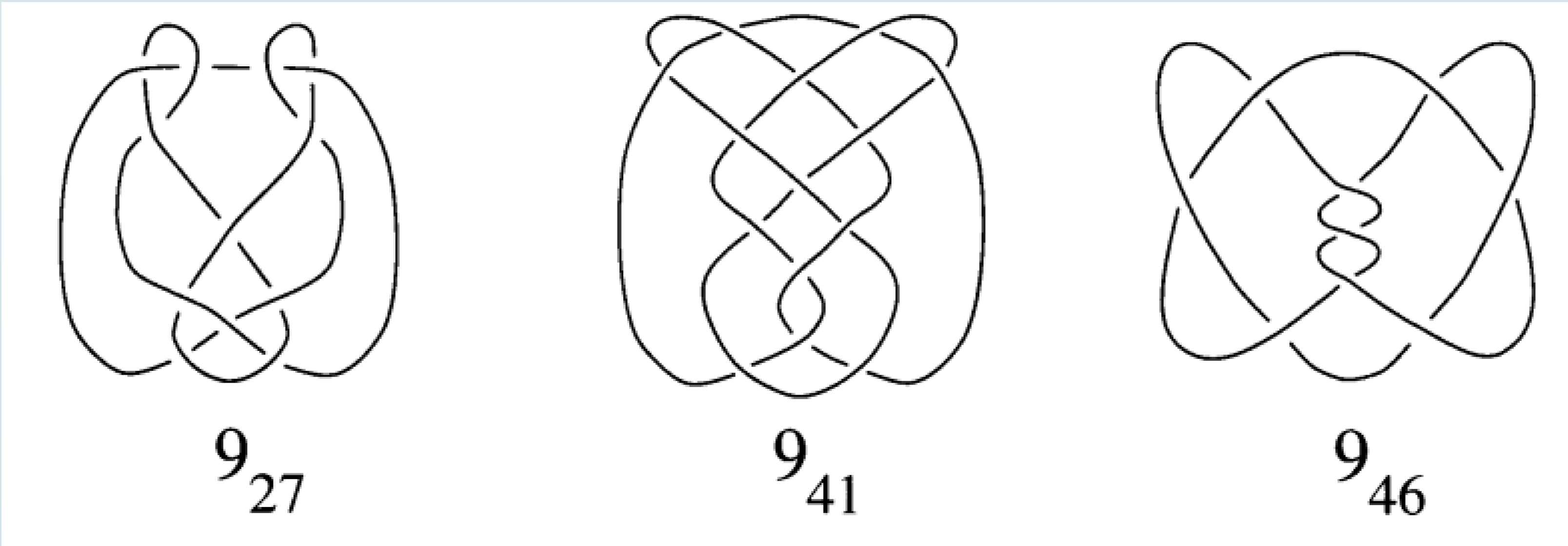
(Eisermann & Lamm, 2008, fig.4)

Main Area of Investigation

The main area of study worked at decomposing a knot to its graph and then examining its spanning trees. The knot is decomposed using the resolution convention, and then to each section of the resulting image we associate a vertex of our graph. The edges in the graph represent the crossings between each section on the original graph. The avenues of investigation were the following:

- number of symmetric spanning trees
- number of total spanning trees
- number of pairs of asymmetric spanning trees
- the sum of the writhes on symmetric spanning trees
- number of vertices
- number of edges

It was thought that we may be more likely to find an invariant under Lamm's symmetric Reidemeister moves, as opposed to only looking at the traditional avenue of the original moves.



(Lamm, 2000, fig.16)

Results from Main Investigation

From analysing the symmetric unions in Lamm's paper, it was revealed there was a trend of graphs producing no symmetric spanning trees. The common denominator amongst these graphs was the appearance of a doubled vertex on the axis of symmetry which produced a loop of sorts, meaning no symmetric spanning trees were possible. From the application of symmetric Reidemeister moves to a handful of Lamm's symmetric unions, this remained invariant under all moves except (S3).

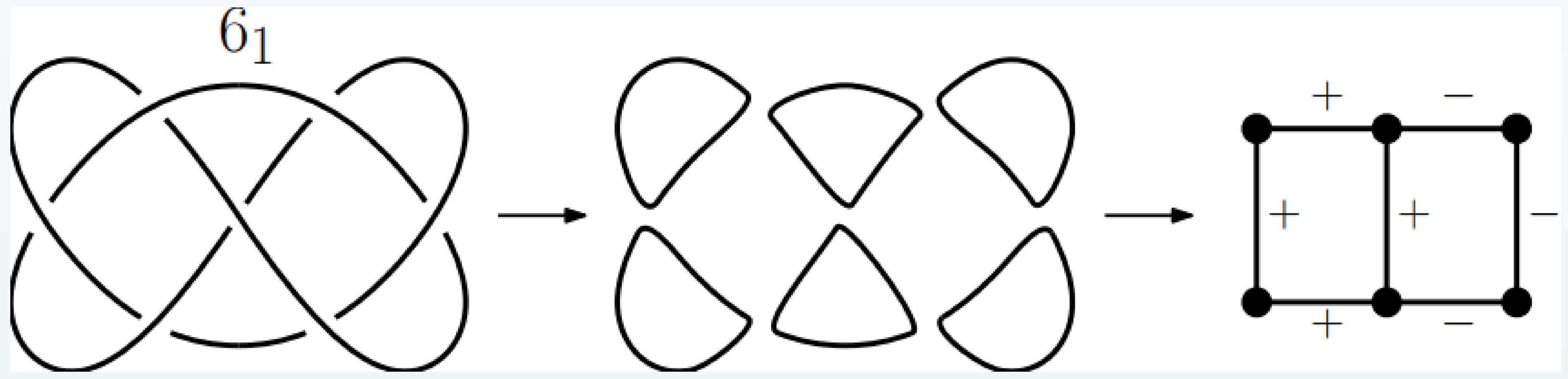


Fig.1 – decomposition of a symmetric union diagram into its graph

Conclusion, Further Investigation, and Acknowledgement

The next avenue of search for the invariant is with quotients and Betti numbers of spanning trees. An issue with loops and invariance has been raised, and future research would look to eliminate this problem. The target invariant would most likely be a polynomial involving a summation of some function of the spanning trees; it is thought that including a component involving the number of horizontal axial crossings may be the solution to the loop issue. I would like to thank my supervisor Dr. William Rushworth for his continued extensive support, and especially for the reformulation of figures (1-3) . This project would not have been possible without him. I would also like to thank the Research Scholarship for funding this project.

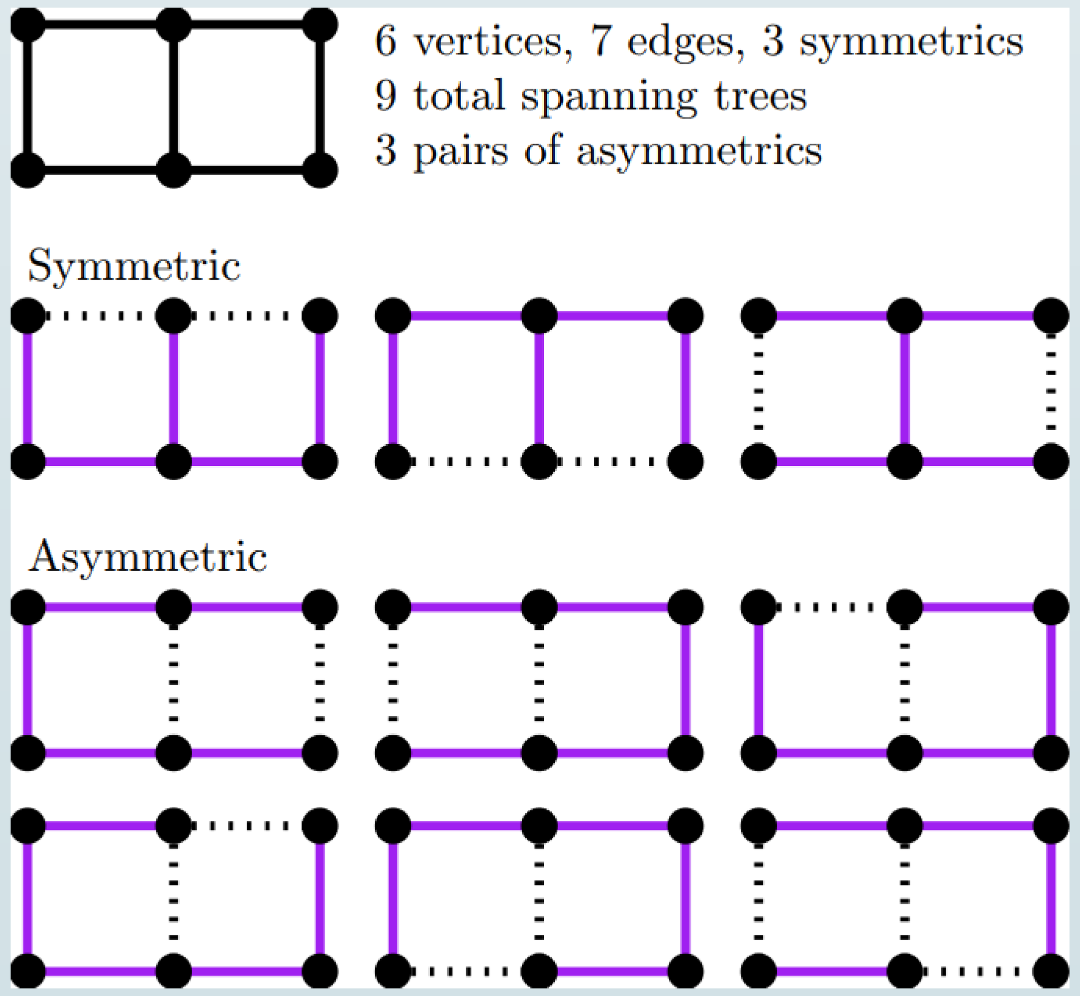


Fig.2 – spanning tree analysis example for 6_1

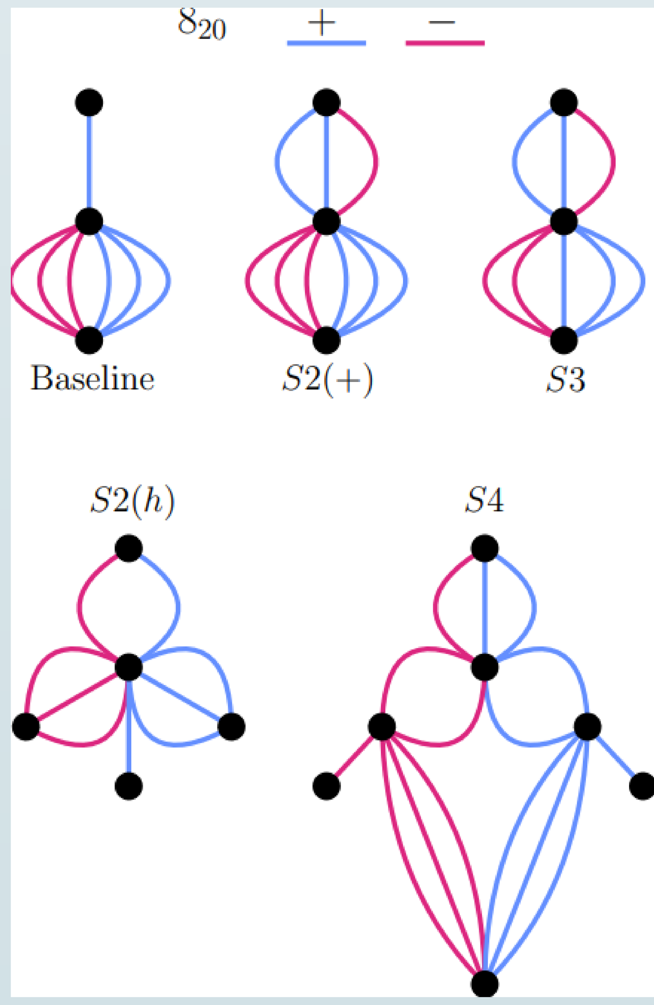


Fig.3 – symmetric RM for 8_{20}

References

- Eisermann, M. and Lamm, C. (2008). A refined Jones polynomial for symmetric unions. *arXiv (Cornell University)*. doi:https://doi.org/10.48550/arxiv.0802.2283.
- Lamm, C. (1998). Symmetric Unions and Ribbon Knots. *Osaka Journal of Mathematics*, 55(2), pp.537–550.
- Thistlethwaite, M.B. (1987). A spanning tree expansion of the jones polynomial. *Topology*, 26(3), pp.297–309. doi:https://doi.org/10.1016/0040-9383(87)90003-6.